LEXSEGMENT IDEALS ARE SEQUENTIALLY COHEN-MACAULAY

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ABSTRACT. The associated primes of an arbitrary lexsegment ideal $I \subseteq S = K[x_1, \ldots, x_n]$ are determined. As application it is shown that S/I is a pretty clean module, therefore, S/I is sequentially Cohen-Macaulay and satisfies Stanley's conjecture.

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1. Introduction

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K. We consider the lexicographical order on the monomials of S induced by $x_1 > x_2 > \ldots > x_n$. Let $d \geq 2$ be an integer and $\mathcal{M}_d$ the set of monomials of degree d of S. For two monomials $u, v \in \mathcal{M}_d$, with $u \geq_{lex} v$, the set

$$L(u, v) = \{w \in \mathcal{M}_d \mid u \geq_{lex} w \geq_{lex} v\}$$

is called a *lexsegment set*. A *lexsegment ideal* in S is a monomial ideal of S which is generated by a lexsegment set. Lexsegment ideals have been introduced by Hulett and Martin [5]. Arbitrary lexsegment ideals have been studied by A. Aramova, E. De Negri, and J. Herzog in [1] and [3]. They characterized all the lexsegment ideals which have a linear resolution. In [4] it was proved that a lexsegment ideal has a linear resolution if and only if it has linear quotients. In the same paper, for a lexsegment ideal $I \subseteq S$, the dimension and the depth of S/I are computed and all the lexsegment ideals which are Cohen-Macaulay are characterized. In [2], the study of the associated prime ideals of a lexsegment ideal is proposed. We answer to this question in Section 2. As an application, by extending a few results from [7] to the multigraded modules over S, we show in Section 3 that S/I is a pretty clean S-module for a lexsegment ideal $I \subseteq S$ (Theorem 3.5). Consequently, it follows that S/I is sequentially Cohen-Macaulay (Corollary 3.8) and the Stanley conjecture ([8]) holds for S/I (Corollary 3.9).

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2. The associated primes of a lexsegment ideal

Let $u = x_1^{a_1} \cdots x_n^{a_n}, v = x_1^{b_1} \cdots x_n^{b_n} \in S$ be two monomials of degree d such that $u \geq_{\text{lex}} v$ and $I = (L(u,v))$ the lexsegment ideal determined by u and v. It is obviously that we may consider $a_1 > 0$ since otherwise we simply study our ideal in a polynomial ring with a smaller number of variables. In addition, we exclude the trivial cases $u = v$ and $I = (L(x_1^d, x_n^d))$. Moreover, we also notice that one may reduce to $b_1 = 0$, that is v is of the form $v = x_q^{b_q} \cdots x_n^{b_n}$ with $q \geq 2$ and $b_q > 0$. Indeed, if $b_1 > 0$, then, from the exact sequence of multigraded S-modules

$$0 \to \frac{S}{(I : x_1^{b_1})} \to \frac{S}{I} \to \frac{S}{(I, x_1^{b_1})} = \frac{S}{(x_1^{b_1})} \to 0, \tag{2.1}$$

we get

$$\operatorname{Ass}(S/(I : x_1^{b_1})) \subseteq \operatorname{Ass}(S/I) \subseteq \operatorname{Ass}(S/(I : x_1^{b_1})) \cup \{(x_1)\}.$$

As $(x_1) \in \operatorname{Ass}(S/I)$ since it is a minimal prime of I, we have $\operatorname{Ass}(S/I) = \operatorname{Ass}(S/(I : x_1^{b_1})) \cup \{(x_1)\}$. Therefore, in order to determine the associated primes of I, we need to compute the associated primes of $(I : x_1^{b_1})$ which is a lexsegment ideal generated in degree $d - b_1$ whose right end, $v/x_1^{b_1}$, is no longer divisible by x_1.

To begin with, we consider two important particular classes, namely, initial and final lexsegment ideals. We recall that a lexsegment ideal of the form $(L(x_1^d, v))$, $v \in \mathcal{M}_d$, is called an *initial lexsegment ideal* determined by v. We denote it by $(L^i(v))$. An ideal generated by a lexsegment set of the form $L(u, x_n^d)$ is called a *final lexsegment ideal* determined by $u \in \mathcal{M}_d$. We denote such an ideal by $(L^f(u))$. We also recall the following notations. For a monomial $w \in S$, we denote $\min(w) = \min\{i : x_i | w\}$, $\max(w) = \max\{i : x_i | w\}$, and $\operatorname{supp}(w) = \{i : x_i | w\}$. In our study we are going to use very often the following

Lemma 2.1. *Let $I = (L(u,v))$ be a lexsegment ideal with $x_1 | u$, $x_1 \nmid v$ and $v \neq x_n^d$. Then*

$$\{(x_1, \ldots, x_j) : j \in \operatorname{supp}(v), j \neq n\} \subseteq \operatorname{Ass}(S/I).$$

Proof. For $j \in \operatorname{supp}(v) \setminus \{n\}$ let $w = (v/x_j) x_n^{d-b_n}$. We can conclude that $w \notin I$. Indeed, if $w \in I$, then $w = m \cdot m'$ for some monomial $m \in L(u,v)$ and m', we get $w \geq_{lex} vm'$ which yields $x_n^{d-b_n} \geq_{lex} x_j m'$, which is impossible. For all $2 \leq i \leq j$, $x_i w = (x_i v/x_j) x_n^{d-b_n} \geq_{lex} v x_n^{d-b_n}$ and $x_1 \nmid (x_i v/x_j) x_n^{d-b_n}$, we have $x_i w = (x_i v/x_j) x_n^{d-b_n} \in I$. Since $x_1 x_n^{d-1} \in I$, it follows that $x_1 w = x_1 (v/x_j) x_n^{d-b_n} = (v/(x_j x_n^{b_n - 1}))(x_1 x_n^{d-1}) \in I$. Therefore $(x_1, \ldots, x_j) \subseteq I : w$. Let us assume that there exists a monomial $z \in I : w$ with $z \notin (x_1, \ldots, x_j)$, that is, $\operatorname{supp}(z) \subseteq \{j+1, \ldots, n\}$ and $wz \in I$. Let $m \in L(u,v)$ such that $wz = mm'$ for some monomial m'. Then we get $v x_n^{d-b_n} z = x_j m m' \geq_{lex} x_j v m'$, which gives $z x_n^{d-b_n} \geq_{lex} x_j m'$ which is contradict with $\operatorname{supp}(z) \subseteq \{j+1, \ldots, n\}$. We thus have shown that $I : w = (x_1, \ldots, x_j)$, which implies that $(x_1, \ldots, x_j) \in \operatorname{Ass}(S/I)$. □

Proposition 2.2. *Let $v \in \mathcal{M}_d$ be a monomial and let $I = (L^i(v))$ the initial ideal determined by v. Then*

$$\operatorname{Ass}(S/I) = \{(x_1, \ldots, x_j) : j \in \operatorname{supp}(v) \cup \{n\}\}.$$

Proof. As we have observed before, we can assume that $v = x_q^{b_q} \cdots x_n^{b_n}$ with $q \geq 2$ and $b_q > 0$. By Lemma 2.1 and [4, Proposition 3.2] we have $\{(x_1, \ldots, x_j) : j \in \mathrm{supp}(v) \cup \{n\}\} \subseteq \mathrm{Ass}(S/I)$.

Let $P \in \mathrm{Ass}(S/I)$, $P \neq (x_1, \ldots, x_n)$. By [6, Proposition 4.2.9] we have $P = (x_1, \ldots, x_j)$, for some $1 \leq j < n$. We want to show that $j \in \mathrm{supp}(v)$. Let us assume $j \notin \mathrm{supp}(v)$. Since $P \supseteq I \supseteq (x_1^d, \ldots, x_q^d)$ it follows that $j > q$. Let w be a monomial such that $w \notin I$ and $P = I : w$. We have $x_j w \in I$, hence there exists $u' \geq_{lex} v$ such that $x_j w = u'm$, for some monomial m. We have $x_j \nmid m$ since, otherwise, $w \in I$. For any $i < j$, we have $x_i \nmid m$ since, otherwise, $x_i u'/x_j >_{\mathrm{lex}} u' \geq_{\mathrm{lex}} v$, and $w = \frac{x_i u'}{x_j} \cdot \frac{m}{x_i} \in I$, contradiction. Therefore, m is a monomial in $K[x_{j+1}, \ldots, x_n]$. We can conclude that $\min(\mathrm{supp}(u')) \geq q$. If there exists $i \leq q-1$ such that $x_i | u'$, then, for any l such that $x_l | m$, we have $i < q < j < l$. Since $\min(\mathrm{supp}(u')) \geq q$, we have $(u'/x_j)x_l >_{lex} v$ by the definition of lexicographical order. Hence $w = (\frac{u'}{x_j} x_l) \cdot \frac{m}{x_l} \in I$, contradiction again. That is u' is of the form

$$u' = x_q^{c_q} \cdots x_n^{c_n} \geq_{lex} x_q^{b_q} \cdots x_n^{b_n} \tag{2.2}$$

If there exists l such that $x_l | m$ and $u' x_l / x_j \geq_{\mathrm{lex}} v$, then as above, $w \in I$, a contradiction. Therefore we must have

$$u' x_l < x_j v \text{ for all } l \text{ such that } x_l | m. \tag{2.3}$$

Using (2.2), and (2.3) and $j \notin \mathrm{supp}(v)$ and by comparing the exponents in the monomials u' and v, we get $u' = x_q^{b_q} \cdots x_{j-1}^{b_{j-1}} x_j x_{j+1}^{c_{j+1}} \ldots x_n^{c_n}$, for some $c_{j+1}, \ldots, c_n$, hence

$$w = x_q^{b_q} \cdots x_{j-1}^{b_{j-1}} x_{j+1}^{c_{j+1}} \cdots x_n^{c_n} \cdot m$$

with $m \in K[x_{j+1}, \ldots, x_n]$. Since $\frac{v}{\gcd(v,w)} \in I : w$, we must have $\frac{v}{\gcd(v,w)} \in (x_1, \ldots, x_j)$, which is impossible since $x_j \nmid v$ and $x_q^{b_q} \cdots x_{j-1}^{b_{j-1}} | \gcd(v, w)$. □

In the next step, we consider final lexsegment ideals. First of all we observe that one should consider only final lexsegment ideals defined by a monomial $u \in \mathcal{M}_d$ such that $x_1 | u$. Indeed, otherwise, we are reduced to considering the problem in a polynomial ring with a smaller number of variables, namely $K[x_{\min(u)}, \ldots, x_n]$.

Proposition 2.3. *Let $u \in \mathcal{M}_d, u \neq x_1^d$, with $x_1 | u$ and $I = (L^f(u))$ be the final lexsegment ideal defined by u. Then*

$$\mathrm{Ass}(S/I) = \{(x_1, \ldots, x_n), (x_2, \ldots, x_n)\}.$$

Proof. By [4, Proposition 3.2], we have $\mathrm{depth}(S/I) = 0$, hence $(x_1, \ldots, x_n) \in \mathrm{Ass}(S/I)$. On the other hand, for any $P \in \mathrm{Ass}(S/I)$, we have $(x_2, \ldots, x_n) \subseteq P$ since $I \supseteq (x_2, \ldots, x_n)^d$. Since $(x_2, \ldots, x_n)$ is obviously a minimal prime of I, we have $(x_2, \ldots, x_n) \in \mathrm{Ass}(S/I)$. Therefore, the only associated primes of I are $\mathfrak{m} = (x_1, \ldots, x_n)$ and $(x_2, \ldots, x_n)$. □

In order to compute the associated primes of an arbitrary lexsegment ideal, that is, one which is neither initial nor final, we are going to distinguish several cases, depending on the depth of S/I. We recall that, by [4, Proposition 3.2], $I = (L(u, v))$ has $\mathrm{depth}(S/I) = 0$ if and only if $x_n u \geq_{\mathrm{lex}} x_1 v$.

Proposition 2.4. *Let $I = (L(u, v))$ be a lexsegment ideal which is neither initial nor final, with $x_1 \nmid v$, and such that* $\operatorname{depth}(S/I) = 0$ *Then*

$$\operatorname{Ass}(S/I) = \{(x_1, \ldots, x_j) : j \in \operatorname{supp}(v) \cup \{n\}\} \cup \{(x_2, \ldots, x_n)\}.$$

Proof. Since $u \neq x_1^d$, we have $I = (I, x_1^{a_1}) \cap (I : x_1^{a_1})$. We get the following exact sequence of S-modules:

$$0 \longrightarrow S/I \longrightarrow S/(I, x_1^{a_1}) \oplus S/(I : x_1^{a_1}) \longrightarrow S/((I, x_1^{a_1}) + (I : x_1^{a_1})) \longrightarrow 0 \tag{2.4}$$

We note that $(I, x_1^{a_1}) + (I : x_1^{a_1}) = (x_1^{a_1}) + (I : x_1^{a_1})$. We first prove $\operatorname{Ass}\left(S/((I, x_1^{a_1}) + (I : x_1^{a_1}))\right) = \{(x_1, \ldots, x_n)\}$ and $\operatorname{Ass}\left(S/(I : x_1^{a_1})\right) = \{(x_2, \ldots, x_n)\}$. If $a_1 > 1$, then $I : x_1^{a_1} \supseteq (x_2, \ldots, x_n)^{d-a_1+1}$, hence $(I, x_1^{a_1}) + (I : x_1^{a_1})$ is an $\mathfrak{m}$-primary monomial ideal, where $\mathfrak{m} = (x_1, \ldots, x_n)$ and $\operatorname{Ass}\left(S/(I : x_1^{a_1})\right) = \{(x_2, \ldots, x_n)\}$. Let $a_1 = 1$. Then we show that $(I : x_1) \supseteq (x_2, \ldots, x_n)^d$, which will imply again that $(I, x_1^{a_1}) + (I : x_1^{a_1}) = (I, x_1) + (I : x_1)$ is $\mathfrak{m}$-primary. Since all the monomials w of degree d with $x_2^d \geq_{\text{lex}} w \geq_{\text{lex}} v$ are already contained in I, thus in $I : x_1$ as well. Hence, we only need to show that $L^f(v) \subseteq (I : x_1)$. Let us assume that there exists a monomial w of degree d with $w <_{lex} v$ such that $w \notin (I : x_1)$, then $x_1 w <_{lex} x_1 v \leq_{lex} x_n u$. As $x_1 | \frac{x_1 w}{x_{\min(w)}}$, $x_1 \nmid v$, we have $\frac{x_1 w}{x_{\min(w)}} >_{lex} v$. By $x_1 \notin (I : x_1)$, we have $\frac{x_1 w}{x_{\min(w)}} >_{lex} u$. Therefore, $w \geq_{lex} \frac{x_n w}{x_{\min(w)}} >_{lex} \frac{x_n u}{x_1} \geq_{lex} v$, where the last inequality follows from the condition $\operatorname{depth}(S/I) = 0$. But then we get $w \geq_{lex} v$, a contradiction. Consequently, we have shown that

$$\operatorname{Ass}\left(S/((I, x_1^{a_1}) + (I : x_1))\right) = \{(x_1, \ldots, x_n)\}$$

and $\operatorname{Ass}(S/(I : x_1)) = \{(x_2, \ldots, x_n)\}$. Since $\operatorname{depth}(S/I) = 0$, hence $\mathfrak{m} \in \operatorname{Ass}(S/I)$, by using the exact sequence (2.4), we get

$$\begin{aligned} \operatorname{Ass}(S/I) = \operatorname{Ass}(S/(I, x_1^{a_1})) \cup \operatorname{Ass}(S/(I : x_1^{a_1})) = \\ = \operatorname{Ass}(S/(I, x_1^{a_1})) \cup \{(x_2, \ldots, x_n)\} \end{aligned} \tag{2.5}$$

Let us first take $a_1 = 1$. It is clear that $P \in \operatorname{Ass}_S(S/(I, x_1))$ if and only if $P = (x_1, P')$, where $P' \in \operatorname{Ass}_{S'}(S'/(L^i(v)))$, where $S' = K[x_2, \ldots, x_n]$. By using Proposition 2.2, we get $\operatorname{Ass}(S/(I, x_1)) = \{(x_1, \ldots, x_j) : j \in \operatorname{supp}(v) \cup \{n\}\}$ and our proof is completed in this case.

Let $a_1 > 1$. Then we consider the exact sequence of S-modules:

$$0 \longrightarrow (I, x_1)/(I, x_1^{a_1}) \longrightarrow S/(I, x_1^{a_1}) \longrightarrow S/(I, x_1) \longrightarrow 0. \tag{2.6}$$

Since $x_1^{d-1}(I, x_1) \subseteq (I, x_1^{a_1})$ and $(x_2, \ldots, x_n)^{d-1}(I, x_1) \subseteq I \subseteq (I, x_1^{a_1})$, it follows that $\operatorname{Ann}_S((I, x_1)/(I, x_1^{a_1}))$ contains an $\mathfrak{m}$-primary ideal, thus we have

$$\operatorname{Ass}((I, x_1)/(I, x_1^{a_1})) = \{\mathfrak{m}\}.$$

From the exact sequence (2.6) and using the above computation for $\operatorname{Ass}(S/(I, x_1))$, we obtain $\mathfrak{m} \in \operatorname{Ass}(S/(I, x_1^{a_1}))$ and $\operatorname{Ass}(S/(I, x_1^{a_1})) \subseteq \{(x_1, \ldots, x_j) : j \in \operatorname{supp}(v) \cup \{n\}\}$. The equality follows by Lemma 2.1. Finally, by using (2.5), we complete the proof. $\square$

We now pass to the case $\operatorname{depth}(S/I) > 0$ which is equivalent to the inequality $x_n u <_{\rm lex} x_1 v$. In particular, this implies that $\deg_{x_l}(u) = 1$. Let $u = x_1 x_l^{a_l} \cdots x_n^{a_n}$ with $l \geq 2$ and $a_l > 0$. The inequality $x_n u <_{\rm lex} x_1 v$ is equivalent to $x_l^{a_l} \cdots x_n^{a_n+1} <_{\rm lex} x_q^{b_q} \cdots x_n^{b_n}$. Therefore we have $l \geq q$. For the next result we introduce the following notation. For $2 \leq j, t \leq n$ such that $2 \leq j \leq t-2$, we denote $P_{j,t} = (x_2, \ldots, x_j, x_t, \ldots, x_n)$.

Proposition 2.5. *Let $I = (L(u,v))$ be a lexsegment ideal with $x_1 \nmid v$ and such that* $\operatorname{depth}(S/I) > 0$.

(i) *Let* $\operatorname{depth}(S/I) = 1$. *Then,*

(a) *for $a_l < d-1$, we have*

$$\operatorname{Ass}(S/I) = \{(x_2, \ldots, x_n)\} \cup \{(x_1, \ldots, x_j) : j \in \operatorname{supp}(v) \setminus \{n\}\} \cup$$

$$\cup \{P_{j,l} : j \in \operatorname{supp}(v), j \leq l-2\} \cup \{P_{j,l+1} : j \in \operatorname{supp}(v), j \leq l-1\};$$

(b) *for $a_l = d-1$, we have*

$$\operatorname{Ass}(S/I) = \{(x_2, \ldots, x_n)\} \cup \{(x_1, \ldots, x_j) : j \in \operatorname{supp}(v) \setminus \{n\}\} \cup$$

$$\cup \{P_{j,l} : j \in \operatorname{supp}(v), j \leq l-2\}.$$

(ii) *Let* $\operatorname{depth}(S/I) > 1$. *Then*

(a) *for $a_l < d-1$, we have* $\operatorname{Ass}(S/I) =$

$$\{(x_1, \ldots, x_j) : j \in \operatorname{supp}(v) \setminus \{n\}\} \cup \{P_{j,l} : j \in \operatorname{supp}(v)\} \cup \{P_{j,l+1} : j \in \operatorname{supp}(v)\};$$

(b) *for $a_l = d-1$, we have*

$$\operatorname{Ass}(S/I) = \{(x_1, \ldots, x_j) : j \in \operatorname{supp}(v) \setminus \{n\}\} \cup \{P_{j,l} : j \in \operatorname{supp}(v)\}.$$

Proof. Since $\operatorname{depth}(S/I) > 0$, we have $\mathfrak{m} \notin \operatorname{Ass}(S/I)$ and $a_1 = 1$, then $(I : x_1) \subseteq (x_2, \ldots, x_n)$. Hence, $\mathfrak{m} \notin \operatorname{Ass}(S/(I : x_1))$ from the exact sequence (2.4), where $a_1 = 1$, we get

$$\operatorname{Ass}(S/I) \subseteq (\operatorname{Ass}(S/(I, x_1)) \setminus \{\mathfrak{m}\}) \cup \operatorname{Ass}(S/(I : x_1)).$$

As in the the proof of Proposition 2.4, we have

$$\operatorname{Ass}(S/(I, x_1)) \setminus \{\mathfrak{m}\} = \{(x_1, \ldots, x_j) : j \in \operatorname{supp}(v) \setminus \{n\}\}.$$

Let us first look at $\operatorname{Ass}(S/(I : x_1))$. Note that $(I : x_1) = J + L$ where J is generated in degree $d-1$ by the final lexsegment $L^f(u/x_1)$, and L is generated in degree d by the initial lexsegment $L^i(v) \subseteq S' = K[x_2, \ldots, x_n]$. Let us first consider $a_l < d-1$. Then, by Proposition 2.3, the associated primes of J are $P_1 = (x_l, \ldots, x_n)$ and $P_2 = (x_{l+1}, \ldots, x_n)$. Therefore, $J = Q_1 \cap Q_2$, where Q_1 and Q_2 are primary monomial ideals with $\sqrt{Q_i} = P_i$, $i = 1, 2$. Similarly, we have $L = \bigcap_{2 \leq j \in \operatorname{supp}(v) \cup \{n\}} Q'_j$ for some monomial primary ideals Q'_j such that $\sqrt{Q'_j} = (x_2, \ldots, x_j)$ for all j. Then

$$(I : x_1) = Q_1 \cap Q_2 + \bigcap_j Q'_j = (\bigcap_j (Q_1 + Q'_j)) \bigcap (\bigcap_j (Q_2 + Q'_j))$$

is a primary decomposition of $I : x_1$. Therefore, by the primary decomposition of $I : x_1$ and $\mathfrak{m} \notin \operatorname{Ass}(S/(I : x_1))$, we get

$$\operatorname{Ass}(S/(I : x_1)) \subseteq \{(x_2, \ldots, x_n)\} \cup \{P_{j,l} : j \in \operatorname{supp}(v), j \leq l-2\} \cup \\ \cup \{P_{j,l+1} : j \in \operatorname{supp}(v), j \leq l-1\}.$$

If $a_l = d-1$, that is $u = x_1 x_l^{d-1}$, then we get that $J = (x_l, \ldots, x_n)^{d-1}$, hence it is a primary ideal. As before, we get

$$\operatorname{Ass}(S/(I : x_1)) \subseteq \{(x_2, \ldots, x_n)\} \cup \{P_{j,l} : j \in \operatorname{supp}(v), j \leq l-2\}.$$

In order to prove (i), taking into account Lemma 2.1, we only need to show that $P_{j,l}, j \leq l-2$, $P_{j,l+1}, j \leq l-1$, and $(x_2, \ldots, x_n)$ are associated primes of I. In each case, we are going to show that one may find a monomial $f \notin I$ such that $I : f = P_{j,l}$ or $P_{j,l+1}$ or $(x_2, \ldots, x_n)$. We begin by proving that $(x_2, \ldots, x_n)$ is an associated prime of I. By [4, Proposition 3.4], $\operatorname{depth}(S/I) = 1$ if and only if $v = x_2^{d-1} x_j$ for some $2 \leq j \leq n-2$ and $j \geq l-1$ or $v \leq_{\text{lex}} x_2^{d-1} x_{n-1}$. If $v \leq_{\text{lex}} x_2^{d-1} x_n$, then, for $f = x_2^{d-1}$, we easily get $I : f = (x_2, \ldots, x_n)$ since all the monomials $x_2^d, x_2^{d-1} x_3, \ldots, x_2^{d-1} x_n$ belong to I. Let $v \geq_{\text{lex}} x_2^{d-1} x_{n-1}$. If $l = 2$, then we choose $f = x_1 x_n^{d-2}$ and observe that $x_1 x_2 x_n^{d-2}, x_1 x_3 x_n^{d-2} \ldots, x_1 x_n^{d-1} \in I$, hence $I : f = (x_2, \ldots, x_n)$. Finally, for $l \geq 3$, we take $f = x_1 x_2^{d-1} x_n^{d-2}$ and get again the desired claim since $x_1 x_l x_n^{d-2}, x_1 x_{l+1} x_n^{d-2} \ldots, x_1 x_n^{d-1}, x_2^d, x_2^{d-1} x_3, \ldots, x_2^{d-1} x_{l-1} \in I$. Therefore, $(x_2, \ldots, x_n) \in \operatorname{Ass}(S/I)$ for $\operatorname{depth}(S/I) = 1$. Now let $j \in \operatorname{supp}(u)$ with $j \leq l-2$, we look for a monomial $f \notin I$ such that $I : f = P_{j,l}, j \leq l-2$. Let us take

$$f = x_1 x_q^{b_q} \cdots x_{j-1}^{b_{j-1}} x_j^{b_j - 1} \cdots x_{l-2}^{b_{l-2}} x_{l-1}^{d} x_l^{a_l - 1} x_{l+1}^{a_{l+1}} \cdots x_n^{a_n}.$$

As $j \in \operatorname{supp}(v)$ and $j \leq l-2$, we have $q \leq j \leq l-2 < l$, then $f >_{lex} u$. Hence $f \notin I$. We now show that $I : f = P_{j,l}$. Let $s \in \{2, \ldots, j, l, \ldots, n\}$. If $s \leq j$, then $x_s f = x_s (v'/x_j) m_1$, where $v' = x_q^{b_q} \cdots x_j^{b_j} \cdots x_{l-2}^{b_{l-2}} x_{l-1}^{d-(b_q + \cdots + b_{l-2})} \geq_{lex} v$ and m_1 is a monomial in S. Since $x_s(v'/x_j) \in L(u,v)$, we get $x_s f \in I$. Let $s \geq l$. Then $x_s f = x_s (u/x_l) m_2$ for some monomial m_2, and since $x_s(u/x_l) \in L(u,v)$, we obtain $x_s f \in I$. We thus showed that $P_{j,l} \subseteq I : f$ for $j \leq l-2$. Let us assume that $P_{j,l} \subsetneqq I : f$, hence there exists a monomial $w \in I : f$ such that $\operatorname{supp}(w) \subseteq \{j+1, \ldots, l-1\}$, that is $w = x_{j+1}^{c_{j+1}} \ldots x_{l-1}^{c_{l-1}}$, where $c_{j+1}, \ldots, c_{l-1} \geq 0$. But

$$wf = x_1 x_q^{b_q} \cdots x_j^{b_j - 1} x_{j+1}^{c'_{j+1}} \cdots x_{l-1}^{c'_{l-1}} x_l^{a_l - 1} x_{l+1}^{a_{l+1}} \cdots x_n^{a_n},$$

and, with same arguments as above, $wf \notin I$. Therefore, $I : f = P_{j,l}$.

Now, let $a_l < d-1$. We show that $P_{j,l+1} \in \operatorname{Ass}(S/I)$ for $j \leq l-1$. If $u = x_1 x_l^{a_l} x_{l+1}^{d-a_l-1}$, we take $f = x_1 x_q^{b_q} \cdots x_{j-1}^{b_j - 1} \cdots x_{l-1}^{b_{l-1}} x_l x_{l+1}^{d-a_l-2}$. If $u <_{lex} x_1 x_l^{a_l} x_{l+1}^{d-a_l-1}$, we take

$$f = x_1 x_q^{b_q} \cdots x_{j-1}^{b_{j-1}} x_j^{b_j - 1} \cdots x_{l-2}^{b_{l-2}} x_{l-1}^{b_{l-1}} x_l^{d} x_{l+1}^{d-a_l-1}.$$

With similar arguments as before, we show that $I : f = P_{j,l+1}$ in each case.

(ii). By [4, Proposition 3.4], $\operatorname{depth}(S/I) > 1$ if and only if $v = x_2^{d-1} x_j$, for some $2 \leq j \leq n-2$ and $l \geq j+2$. In this case $(x_2, \ldots, x_n) \notin \operatorname{Ass}(S/I)$ and the conclusion

follows directly from Lemma 2.1 and by looking at $\operatorname{Ass}(S/(I:x_1))$. Indeed, we have the exact sequence

$$0 \longrightarrow S/(I:x_1) \longrightarrow S/I \longrightarrow S/(I,x_1) \longrightarrow 0,$$

thus $\operatorname{Ass}(S/(I:x_1)) \subseteq \operatorname{Ass}(S/I) \subseteq \operatorname{Ass}(S/(I:x_1)) \cup \operatorname{Ass}(S/(I,x_1))$. As $(x_1,\ldots,x_j) \in \operatorname{Ass}(S/I)$ for all $j \in \operatorname{supp}(v)$, $j \neq n$, we only need to compute $\operatorname{Ass}(S/(I:x_1))$. Note that, in this case,

$$(I:x_1) = \begin{cases} (L^f(u/x_1)) + (x_2^d), & \text{if } \ v = x_2^d, \\ (L^f(u/x_1)) + (x_2^{d-1}) \cap (x_2^d, x_3, \ldots, x_j), & \text{if } v = x_2^{d-1}x_j, \\ & \ 3 \leq j \leq n-2, \end{cases}$$

If $v = x_2^d$, we get, by using Proposition 2.3, $(I:x_1) = (x_2^d, Q_1) \cap (x_2^d, Q_2)$ where Q_1, Q_2 are primary ideals with $\sqrt{Q_1} = (x_l, \ldots, x_n)$ and $\sqrt{Q_2} = (x_{l+1}, \ldots, x_n)$, which implies that $\operatorname{Ass}(S/(I:x_1)) = \{P_{2,l}, P_{2,l+1}\}$. Finally, if $v = x_2^{d-1}x_j$, with $3 \leq j \leq n-2$, we get, by using Proposition 2.3,

$$(I:x_1) = (x_2^{d-1}, Q_1) \cap (x_2^{d-1}, Q_2) \cap (x_2^d, x_3, \ldots, x_j, Q_1) \cap (x_2^d, x_3, \ldots, x_j, Q_2),$$

where Q_1, Q_2 are primary and $\sqrt{Q_1} = (x_l, \ldots, x_n)$, $\sqrt{Q_2} = (x_{l+1}, \ldots, x_n)$. This yields $\operatorname{Ass}(S/(I:x_1)) = \{P_{j,l}, P_{j,l+1} : j \in \operatorname{supp}(v)\}$. □

3. Lexsegment ideals are pretty clean

Pretty clean modules were defined in [7]. Since we are interested in finitely generated multigraded modules over S, we recall the definition of pretty cleanness in this frame.

Definition 3.1 ([7])**.** *Let M be a finitely generated multigraded S-module. A multigraded prime filtration of M,*

$$\mathcal{F}: \quad 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{r-1} \subseteq M_r = M,$$

where $M_i/M_{i-1} \cong S/P_i$, with P_i a monomial prime ideal, is called ***pretty clean*** *if for all $i < j$, $P_i \subseteq P_j$ implies $i = j$. In other words, a proper inclusion $P_i \subseteq P_j$ is possible only if $i > j$. A multigraded S-module is called* ***pretty clean*** *if it admits a pretty clean filtration.*

We denote by $\operatorname{Supp}(\mathcal{F})$ the set $\{P_1, \ldots, P_r\}$ of the prime ideals which define the factor modules of $\mathcal{F}$. By [7, Corollary 3.4.], $\operatorname{Supp}(\mathcal{F}) = \operatorname{Ass}(S/I)$.

The following lemma gives a nice class of pretty clean multigraded S-modules.

Lemma 3.2. *Let M be a finitely generated multigraded S-module such that* $\operatorname{Ass}(M)$ *is totally ordered by inclusion. Then M is pretty clean.*

The proof works as the proof of [7, Proposition 5.1], therefore we omit it.

Our aim in this section is to show that if $I \subseteq S$ is a lexsegment ideal, then S/I is pretty clean. The claim is obvious for initial and final lexsegment ideals. Indeed, by applying Proposition 2.2, Proposition 2.3, and the above lemma, we get

Corollary 3.3. *Let $I \subseteq S$ be an initial or final lexsegment ideal. Then S/I is pretty clean.*

For arbitrary lexsegment ideals we need another preparatory result.

Lemma 3.4. *Let* $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ *be an exact sequence of finitely generated multigraded* S*-modules and homogeneous morfisms. We assume that* M' *has a multigraded pretty clean filtration* $\mathcal{F}'$ *and* M'' *has a multigraded pretty clean filtration* $\mathcal{F}''$ *such that for any* $P \in \mathrm{Supp}(\mathcal{F}')$ *and* $Q \in \mathrm{Supp}(\mathcal{F}'')$*, we have* $P \not\subseteq Q$*, that is either* $P \supseteq Q$ *or* P *and* Q *are incomparable by inclusion. Then* M *is pretty clean.*

Proof. Let $\mathcal{F}' : 0 = M'_0 \subseteq \cdots \subseteq M'_r = M'$ be the filtration of M' and $\mathcal{F}'' : 0 = M''_0 \subseteq \cdots \subseteq M''_s = M''$ the filtration of M''. Then, by hypothesis, the following filtration,

$$0 = f(M'_0) \subseteq \cdots \subseteq f(M'_r) = f(M') = g^{-1}(0) \subseteq \cdots \subseteq g^{-1}(M''_s) = M$$

is a multigraded prime filtration of M, hence M is pretty clean. □

The first consequence that one derives from the above lemma is that we can reduce, as in the previous section, to the case when v, the right end of the lexsegment set which generates the lexsegment ideal, is not divisible by x_1. Indeed, if $\deg_{x_1}(v) = b_1 > 0$, looking at the exact sequence (2.1), we see that, in order to prove that S/I is pretty clean, it is enough to show that $S/(I : x_1^{b_1})$ is pretty clean since $\mathrm{Ass}(S/(I : x_1^{b_1}))$ obviously does not contain (x_1).

Theorem 3.5. *Let* $I \subseteq S$ *be a lexsegment ideal. Then* S/I *is a pretty clean module.*

The proof of the theorem will follow from Corollary 3.3 and the next two lemmas. As in the previous section, we consider separately the cases when $\mathrm{depth}(S/I) = 0$ and $\mathrm{depth}(S/I) > 0$.

Lemma 3.6. *Let* I *be a lexsegment ideal which is neither initial nor final and such that* $\mathrm{depth}(S/I) = 0$ *and* $x_1 \nmid v$*. Then* S/I *is pretty clean.*

Proof. Let $u = x_1^{a_1} \ldots x_n^{a_n}$ with $a_1 > 0$ and $x_q^{b_q} \ldots x_n^{b_n}$ with $q \geq 2$ and $b_q > 0$. We consider the exact sequence of multigraded modules:

$$0 \longrightarrow (I : x_1^{a_1})/I \longrightarrow S/I \longrightarrow S/(I : x_1^{a_1}) \longrightarrow 0. \tag{3.1}$$

As $x_1^{a_1} \in \mathrm{Ann}_S((I : x_1^{a_1})/I)$, we get $x_1 \in P$ for all $P \in \mathrm{Ass}((I : x_1^{a_1})/I)$. On the other hand, as we already noticed in the proof of Proposition 2.4, $\mathrm{Ass}(S/(I : x_1^{a_1})) = \{(x_2, \ldots, x_n)\}$. By Proposition 2.4, we have

$$\mathrm{Ass}(S/I) = \{(x_1, \ldots, x_j) : j \in \mathrm{supp}(v) \cup \{n\}\} \cup \{(x_2, \ldots, x_n)\},$$

which implies that $\mathrm{Ass}((I : x_1^{a_1})/I) = \{(x_1, \ldots, x_j) : j \in \mathrm{supp}(v) \cup \{n\}\}$, thus by Lemma 3.2, $(I : x_1^{a_1})/I$ and $S/(I : x_1^{a_1})$ are pretty clean S-modules. Next we apply Lemma 3.4 and conclude that S/I is pretty clean. □

Lemma 3.7. *Let* I *be a lexsegment ideal such that* $\mathrm{depth}(S/I) > 0$ *and* $x_1 \nmid v$*. Then* S/I *is pretty clean.*

Proof. As we have seen before, since $\mathrm{depth}(S/I) > 0$, u and v have the following form: $u = x_1 x_l^{a_l} \cdots x_n^{a_n}$ with $l \geq 2$ and $a_l > 0$, $v = x_q^{b_q} \cdots x_n^{b_n}$ with $q \geq 2$ and $b_q > 0$.

Moreover, we have $l \geq q$. As in the first part of the proof of Lemma 3.6, by using the exact sequence of multigraded S-modules

$$0 \to \frac{(I : x_1)}{I} \to \frac{S}{I} \to \frac{S}{(I : x_1)} \to 0, \tag{3.2}$$

it is enough to show that $(I : x_1)/I$ and $S/(I : x_1)$ are pretty clean and no prime ideal of the pretty clean filtration of $(I : x_1)/I$ is strictly contained in a prime ideal of the pretty clean filtration of $S/(I : x_1)$. We first observe that since $x_1 \in \mathrm{Ann}_S((I : x_1)/I)$, we have $x_1 \in P$ for all the prime ideals $P \in \mathrm{Ass}((I : x_1)/I)$. On the other hand, since x_1 is regular on $S/(I : x_1)$, it follows that $x_1 \notin P$ for all $P \in \mathrm{Ass}(S/(I : x_1))$. By Proposition 2.5, we get $\mathrm{Ass}((I : x_1)/I) = \{(x_1, \ldots, x_j) : j \in \mathrm{supp}(v) \setminus \{n\}\}$, therefore $(I : x_1)/I$ is a pretty clean module since its associated primes are totally ordered by inclusion.

If $u = x_1 x_l^{d-1}$, it follows, by Proposition 2.5, that $\mathrm{Ass}(S/(I : x_1)) \subseteq \{(x_2, \ldots, x_j, x_l, \ldots, x_n) : j \in \mathrm{supp}(v)\} \cup \{(x_2, \ldots, x_n)\}$, thus it is totally ordered by inclusion, which shows that $S/(I : x_1)$ is pretty clean. The same argument works if $u <_{\mathrm{lex}} x_1 x_l^{d-1}$ and $q = l$. In both cases, it is clear that for all $P \in \mathrm{Ass}((I : x_1)/I)$ and $P' \in \mathrm{Ass}(S/(I : x_1))$ we have $P \not\subseteq P'$. We then may conclude that in these cases S/I is a pretty clean module.

It remains to consider $\deg_{x_l}(v) < d - 1$ and $q \leq l - 1$. We are going to show that $S/(I : x_1)$ is pretty clean which will end our proof. Note that one may decompose $(I : x_1)$ as $(I : x_1) = J + L$ where J is generated in degree $d - 1$ by the final lexsegment $L^f(u/x_1) \subseteq K[x_l, \ldots, x_n]$, and L is generated in degree d by the initial lexsegment $L^i(v) \subseteq K[x_2, \ldots, x_n]$. Let $(L^i(v)) = \bigcap_{j \in \mathrm{supp}(v) \cup \{n\}} Q_j$ be the irredundant primary decomposition of $(L^i(v))$ where Q_j are monomial primary ideals with $\sqrt{Q_j} = (x_2, \ldots, x_j), j \in \mathrm{supp}(v) \cup \{n\}$. Let $M = (I : x_1) : x_l^d = (J + (L^i(v))) : x_l^d = J : x_l^d + (L^i(v)) : x_l^d$. It is easily seen that $J : x_l^d$ is a monomial $(x_{l+1}, \ldots, x_n)$-primary ideal. In addition, we have $(L^i(v)) : x_l^d = (\bigcap_{j \in \mathrm{supp}(v) \cup \{n\}} Q_j) : x_l^d = \bigcap_{j \in \mathrm{supp}(v) \cup \{n\}} (Q_j : x_l^d) = (\bigcap_{\substack{j \in \mathrm{supp}(v) \\ j \leq l-1}} (Q_j : x_l^d)) \bigcap (\bigcap_{\substack{j \in \mathrm{supp}(v) \cup \{n\} \\ j \geq l}} (Q_j : x_l^d))$. In the last intersection, each of the primary monomial ideals contains a power of x_l, therefore $Q_j : x_l^d = S$ for all $j \geq l$. It follows that $(L^i(v)) : x_l^d = \bigcap_{\substack{j \in \mathrm{supp}(v) \\ j \leq l-1}} (Q_j : x_l^d)$. This implies that $M = \bigcap_{\substack{j \in \mathrm{supp}(v) \\ j \leq l-1}} (J : x_l^d + Q_j : x_l^d) = \bigcap_{\substack{j \in \mathrm{supp}(v) \\ j \leq l-1}} (J : x_l^d + Q_j)$ is an iredundant primary decomposition of M which gives $\mathrm{Ass}(S/M) = \{(x_2, \ldots, x_j, x_{l+1}, \ldots, x_n) : j \in \mathrm{supp}(v), j \leq l - 1\}$. It is clear that $M \supseteq I : x_1$, hence we have the exact sequence of multigraded S-modules

$$0 \to \frac{M}{(I : x_1)} \to \frac{S}{(I : x_1)} \to \frac{S}{M} \to 0.$$

On the other hand, it is also clear that $x_l^d M \in I : x_1$, which implies that $x_l^d \in \mathrm{Ann}(M/(I : x_1))$. In particular, it follows that $x_l \in P$ for all $P \in \mathrm{Ass}(M/(I : x_1))$. From the above sequence and by using the form of $\mathrm{Ass}(S/(I : x_1))$ we finally get $\mathrm{Ass}(M/(I : x_1)) = \{(x_2, \ldots, x_j, x_l, \ldots, x_n) : j \in \mathrm{supp}(v)\}$, hence $M/(I : x_1)$ is pretty clean. Moreover, there is no proper inclusion of the type $P \subseteq P'$ where

$P \in \operatorname{Ass}(M/(I : x_1))$ and $P' \in \operatorname{Ass}(S/M)$, hence, by Lemma 3.4, $S/(I : x_1)$ is pretty clean. $\square$

Theorem 3.5 and Corollary 4.3. in [7] yield the following

Corollary 3.8. *Let $I \subseteq S$ be a lexsegment ideal. Then S/I is sequentially Cohen-Macaulay.*

Moreover, from Theorem 3.5 and [7, Theorem 6.5.] we get the following

Corollary 3.9. *Let $I \subseteq S$ be a lexsegment ideal. Then S/I satisfies the Stanley conjecture, that is we have the inequality* $\operatorname{sdepth}(S/I) \geq \operatorname{depth}(S/I)$, *where* $\operatorname{sdepth}(S/I)$ *is the Stanley depth of S/I.*

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